ISRAEL JOURNAL OF MATHEMATICS **126** (2001), 263–267

INFINITE $B_2[g]$ SEQUENCES

BY

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ABSTRACT

We exhibit, for any integer $g \geq 2$, an infinite sequence $A \in B_2[g]$ such that $\limsup_{x\to\infty} A(x)/\sqrt{x} = (\sqrt{9/8})\sqrt{g-1}$. Furthermore, we obtain better estimates for small values of g. For instance, we exhibit an infinite sequence $A \in B_2[2]$ such that $\limsup_{x\to\infty} A(x)/\sqrt{x} = \sqrt{3/2}$.

Introduction

For $g \in \mathbb{N}$, denote by $B_2[g]$ the class of all sets $A \subset \mathbb{N}$ such that, for all $n \in \mathbb{N}$, the equation $a + a' = n$, with $a, a' \in A$, $a \leq a'$, has at most g solutions. The sets $B_2[1]$ are called **Sidon sets.**

Erdős proved in [6] that, if A is an infinite Sidon sequence, then $\liminf_{x\to\infty} A(x)/x^{1/2} = 0$, where $A(x) = \#\{a \leq x: a \in A\}$ is the counting function. On the other hand, he showed that there exists an infinite Sidon sequence such that $\limsup_{x\to\infty} A(x)/x^{1/2} = 1/2$. This limit was improved to be $1/\sqrt{2}$ by Kruckeberg [5]. Much less is known on infinite $B_2[g]$ sequences for $g > 1$. It is conjectured that $\liminf_{x\to\infty} A(x)/x^{1/2} = 0$ for any infinite $B_2[g]$ sequence, but this is unproved even for $g = 2$.

* Partially supported by Colciencias, Colombia and Universidad del Cauca.
Received March 7, 1999 and in revised form September 15, 2000

As for lim sup, Kolountzakis [4] proved there is an infinite $B_2[2]$ sequence A such that $\limsup_{x\to\infty} A(x)/x^{1/2} = 1$. Xing De Jia dealt in [3] with such limits. Although his method does not work for usual $B_2[g]$ sequences, he gets upper bounds for sequences such that, for a fixed module m, the number of solutions $a, a' \in A$ for the equation $n \equiv a + a' \pmod m$, $a \le a'$, is at most g, for each integer n.

By proceeding in a different way, we have improved the former lower bounds on infinite $B_2[g]$ sequences for $g \ge 2$:

THEOREM 1: *For all $g \ge 2$ there exists an infinite $B_2[g]$ sequence A such that $\limsup_{x\to\infty} A(x)/\sqrt{x} = L_g$ where*

$$L_g = \begin{cases} \sqrt{3/2}, & g = 2, \\ 3/2, & g = 3, \\ \sqrt{\frac{36}{11}}, & g = 4, \\ \sqrt{\frac{9}{2}}, & g = 5, \\ \sqrt{\frac{100}{17}}, & g = 6, \\ \sqrt{\frac{27}{4}}, & g = 7, \\ \sqrt{8}, & g = 8, \\ \sqrt{\frac{9}{8}}\sqrt{g-1}, & g \ge 9. \end{cases}$$

Proof of Theorem 1

In order to prove the theorem, we only need to show that any $B_2[g]$ sequence

$$A_0 = \{n_1 < n_2 < \cdots < n_k\}$$

can be extended to a $B_2[g]$ sequence

$$A = \{n_1 < \cdots < n_k < n_{k+1} < \cdots < n_l\},$$

where $l/\sqrt{n_l} = A(n_l)/\sqrt{n_l} \ge L_g + o(1)$.

For the construction of A we need two special sets C_g and B_p; their properties are stated in the following two propositions, whose proofs we postpone to the end of the section.

PROPOSITION 1: *For any prime p, there exists a set $B_p \subset (p^{1/2}, p^2 - p^{1/2})$ such that*

(i) *if $b + b' \equiv b'' + b''' \mod p^2 - 1$ for $b, b', b'', b''' \in B_p$, then $\{b, b'\} = \{b'', b'''\}$,*

(ii) *$|b - b'| > p^{1/2}$ for all different $b, b' \in B_p$,*

(iii) $|B_p| > p - 4p^{1/2}$.

PROPOSITION 2: *For $g \geq 2$, there exists an integer u_g and a set $C_g \subset [0, u_g]$ such that $r(n) = \#\{n = c + c'; \quad c, c' \in C_g\}$. Then*

(i) $r(n) \leq g$ for all integer n,

(ii) $r(c) \leq g - 1$ and $r(c-1) \leq g - 1$ for all $c \in C_g$,

(iii) $|C_g|/\sqrt{u_g + 1} = L_g$, where L_g is defined as in Theorem 1.

THE CONSTRUCTION OF A. For $x = n_k$, a prime p such that $x^2 < p < 2x^2$, and $m = p^2 - 1$ we define $A = A_0 \cup D$ where

$$D = \bigcup_{c \in C_g} (B_p + cm + 2x);$$

the sets B_p and C_g are defined as in Propositions 1 and 2.

Obviously, the sequence A is an extension of A_0. Thus, we are supposed to prove that A is a $B_2[g]$ sequence satisfying $|A|/\sqrt{n_l} = L_g + o(1)$, where n_l is the biggest element of A.

PROPOSITION 3: *The sequence A is a $B_2[g]$ sequence.*

Proof: If $n \leq 2x$, then all the representations of n as a sum of two elements of A are of the form $a + a'$ with $a, a' \in A_0$. Since A_0 is a $B_2[g]$ sequence, there are at most g representations.

If $n > 2x$, there is at most one representation of the form $a + d$ with $a \in A_0$, $d \in D$. Otherwise, if $n = a + d = a' + d'$, then $x > |a - a'| = |d - d'| = |b - b' + (c_i - c_{i'})m| > p^{1/2} > x$. We have used in this inequality the property (ii) of Proposition 1 whenever $c_i = c_{i'}$ and the condition $B_p \subset (p^{1/2}, p^2 - p^{1/2})$ whenever $c_i \neq c_{i'}$. We consider two cases:

(1) $n \in A_0 + D$. Suppose there are more than g representations

$$n = a + d = d_1 + d_1' = \cdots = d_g + d_g',$$

$$n = a + (b + cm + 2x)$$

$$= (b_1 + c_{j_1}m + 2x) + (b_1' + c_{j_1'}m + 2x) = \cdots = (b_g + c_{j_g}m + 2x) + (b_g' + c_{j_g'}m + 2x).$$

We can assume that d_i and d_i' are such that $b_i \leq b_i'$ and, because of the property (i) of Proposition 1, we obtain $b_i = b_j$ and $b_i' = b_j'$ for all i, j. Therefore,

$$a + b - b_1 - b_1' - 2x + cm = (c_1 + c_1')m = \cdots = (c_g + c_g')m.$$

We remark that $a + b - b_1 - b_1' - 2x < x + (p^2 - p^{1/2}) - p^{1/2} - p^{1/2} - 2x < p^2 - 3p^{1/2} < m$ and $a + b - b_1 - b_1' - 2x > 1 + p^{1/2} - (p^2 - p^{1/2}) - (p^2 - p^{1/2}) - 2x =$

$-2m+3p^{1/2}-2x-1 > -2m$, where, in the last inequality, we have used $x < p^{1/2}$. As a consequence, $a+b-b_1-b_1'-2x$ is either 0 or $-m$. If we divide all terms in the above inequality by m, we obtain g representations of c or $c-1$ as sums of elements of C_g. But this is impossible, because of property (ii) of Proposition 2.

(2) $n \notin A_0 + D$. Then all the representations are of the form $d+d'$ with $d, d' \in D$. If we order the sets $\{d_i, d_i'\}$ as before, we obtain $b_i = b_j$ and $b_i' = b_j'$ for all i, j. If there were more than g representations, there would be more than g different representations of an integer of the form $c_i + c_i'$ with $c_i, c_i' \in C_g$, which is impossible because of property (i) of Proposition 2. ∎

Proposition 4: *For the biggest element n_l of A, we have*

$$\frac{|A|}{\sqrt{n_l}} = L_g + o(1).$$

Proof: The cardinality of the sequence A is

$$|A| = |A_0| + |C_g||B_p| \geq |C_g| m^{1/2}(1+o(1)),$$

and $A \subset [1, (u_g+1)m + o(m)]$.

As a consequence,

$$\frac{|A|}{\sqrt{n_l}} = \frac{|C_g| m^{1/2}(1+o(1))}{\sqrt{m(u_g+1)+o(m)}} = L_g + o(1),$$

in view of property (iii) of Proposition 2. ∎

Proof of Proposition 1: Chowla and Erdős [2] proved that, for every prime p, there exists a Sidon sequence $B \subset [1, p^2-1]$ with p terms such that, if $b + b' \equiv b'' + b''' \mod p^2 - 1$, then $\{b, b'\} = \{b'', b'''\}$.

The set B_p, for which we are looking, will be the set B minus the subset of elements lying in the union of intervals $[1, p^{1/2}] \cup [p^2 - p^{1/2}, p^2-1]$ and minus those b, b' such that $|b - b'| < p^{1/2}$.

Since all differences $b - b'$ are distinct, the cardinality of $B \setminus B_p$ is at most $4p^{1/2}$. ∎

Proof of Proposition 2: It was proved in [1] that the set

$$A^g = \{k\colon\ \leq k \leq g-1\} \cup \{g-1+2k\colon 1 \leq k \leq [g/2]\}$$

satisfies $r(n) \leq g$, for any integer n.

We take $C_g = A^{g-1}$ and thus $r(n) \leq g-1$, for any integer n. In particular, the set C_g satisfies conditions (i) and (ii) of Proposition 2. It is easy to check that (iii) is also satisfied.

For $g \leq 8$, let

$$C_2 = \{1,2,5\}, \quad C_3 = \{0,1,3\}, \quad C_4 = \{0,1,2,4,7,10\},$$
$$C_5 = \{0,1,2,3,5,7\}, \quad C_6 = \{0,1,2,3,4,6,8,11,13,16\},$$
$$C_7 = \{0,1,2,3,4,5,7,9,11\}, \quad C_8 = \{0,1,2,3,4,5,6,8,10,12,15,17\}.$$

It is easy to check that these sets satisfy the conditions of Proposition 2. ∎

ACKNOWLEDGEMENT: We thank the referee for many suggestions which have made this article more readable.

References

[1] J. Cilleruelo, I. Ruzsa and C. Trujillo, *Upper and lower bound for finite $B_h[g]$ sequences, $g > 1$*, preprint.

[2] H. Halberstam and K. F. Roth, *Sequences*, Springer-Verlag, New York, 1983.

[3] X.-D. Jia, *$B_h[g]$ sequences with large upper density*, Journal of Number Theory **56** (1996), 298–308.

[4] M. N. Kolountzakis, *The density of $B_h[g]$ sequences and the minimun of dense cosine sums*, Journal of Number Theory **56** (1996), 4–11.

[5] F. Kruckeberg, *B_2 Folgen und verwandte Zahlenfolgen*, Journal für die reine und angewandte Mathematik **206** (1961), 53–60.

[6] A. Stöhr, *Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe II*, Journal für die reine und angewandte Mathematik **194** (1955), 111–140.